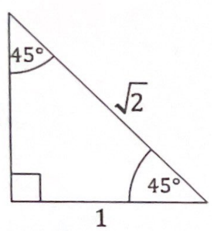
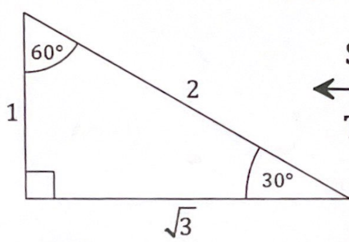
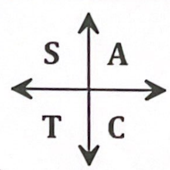






$\sin(0°) = 0$	$\cos(0°) = 1$
$\sin(90°) = 1$	$\cos(90°) = 0$
$\sin\left(\frac{\pi}{2}\right) = 1$	$\cos\left(\frac{\pi}{2}\right) = 0$
$\sin(180°) = 0$	$\cos(180°) = -1$
$\sin(\pi) = 0$	$\cos(\pi) = -1$

$$\theta = \text{rad} \times \frac{180}{\pi}$$

$$\text{rad} = \theta \times \frac{\pi}{180}$$

$$180^\circ = \pi$$

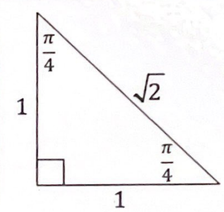
$$360^\circ = 2\pi$$

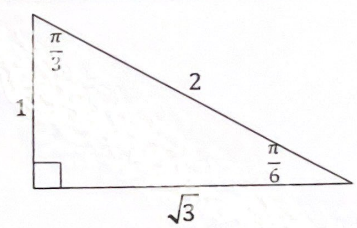
$$90^\circ = \frac{\pi}{2}$$

$$30^\circ = \frac{\pi}{6}$$

$$60^\circ = \frac{\pi}{3}$$

$$45^\circ = \frac{\pi}{4}$$





Kyla Orchard

DIFFERENTIATION

CHAIN RULE

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

if $y = [f(x)]^n$
then dy/dx
$= n[f(x)]^{n-1} \cdot f'(x)$

example $y = (2x-1)^5$

$y = u^5 \quad u = 2x - 1$

$dy/du = 5u^4 \quad du/dx = 2$

$\therefore \frac{dy}{dx} = 5(2x-1)^4(2)$

$= 10(2x-1)^4$

PRODUCT RULE

$y = f(x) \cdot g(x)$

$\frac{dy}{dx} = f'(x) \cdot g(x) + g'(x) \cdot f(x)$

example $y = 3x^2(2x+1)^3$

$y' = 6x(2x+1)^3 + 3x^2 \cdot 3(2x+1)^2(2)$

$= 6x(2x+1)^3 + 18x^2(2x+1)^2$

$= 6x(2x+1)^2[2x+1+3x]$

$= 6x(2x+1)^2(5x+1)$

QUOTIENT RULE

$y = \frac{f(x)}{g(x)}$

$\frac{dy}{dx} = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2}$

example $y = \frac{(x^2+1)}{(1-x)}$

$y' = \frac{(1-x)2x - (x^2+1)(-1)}{(1-x)^2}$

$= \frac{2x - 2x^2 + x^2 + 1}{(1-x)^2}$

$= \frac{1 + 2x - x^2}{(1-x)^2}$

RATES OF CHANGE

V = volume t = time

$dy/dx = 0$ @ t.p's

$d^2y/dx^2 = 0$ @ POI

if dy/dx & $d^2y/dx^2 = 0$
then possible horizontal POI
(use sign test)

ACCELERATION

displacement = $x [= f(t)]$

velocity = dx/dt

acceleration = dv/dt

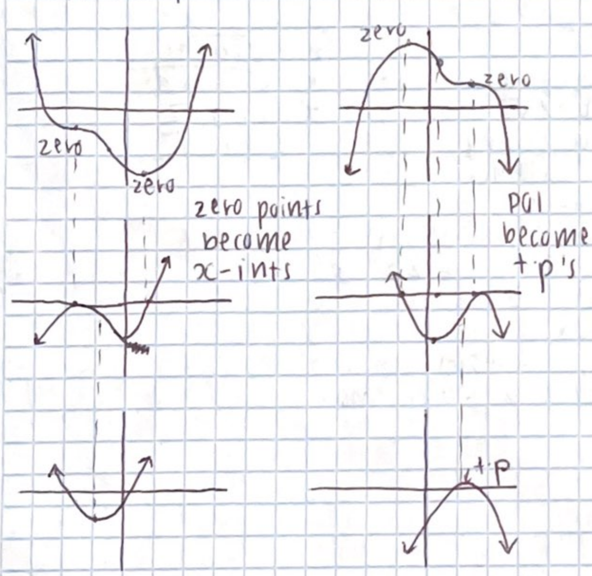
displacement
velocity
acceleration
(require direction)

DOT NOTATION

$dy/dt = \dot{y} \quad v = \dot{x}$

$d^2y/dt^2 = \ddot{y} \quad a = \ddot{x}$

EXAMINING THE 2nd DERIVATIVE



LOCATING STATIONARY POINTS (OPTIMISATION)

1. draw diagram
2. identify variable to be maximised or minimised (e.g. C)
3. if equation for C has 2 variables, find another equation that can sub for one
4. find C in terms of 1 variable
5. find values of x where $dC/dx = 0$
6. use $y''(x)$ test to determine max or min
7. check domain/range + plot

when $f''(x) < 0$ then $f(x)$
= concave down

when $f''(x) > 0$ then $f(x)$
= concave up

when $f''(x) = 0$ then = POI

but $f''(x) = 0$ doesn't always mean it's a POI

EXAMPLE - SOLVING

$y = ax^4 + bx^3 - x^2 + 1$
has POI @ $(1, -4)$
find a & b

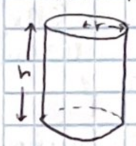
$y' = 4ax^3 + 3bx^2 - 2x$
$y'' = 12ax^2 + 6bx - 2$
$y'' = 0$ @ $x = 1$
$0 = 12a + 6b - 2$
$y = -4$ when $x = 1$
$-4 = a + b \cancel{+1} \cancel{-1}$
$a + b = -4$
then substitution or elimination
$6a + 3b = 1$

elimination:
$-3a - 3b = 12$
$3a = 13$
$a = 13/3$
$\therefore 13/b + b = -4$
$b = -12/3, -13/3$
$b = -25/3$

sub:
$a = -4 - b$
sub back into
$(6a + 3b = 1)$
$6(-4 - b) + 3b = 1$
$-24 = 6b + 3b = 1$
$-25 = 3b$
$b = -25/3$

OPTIMISATION EXAMPLE

closed cylinder $V = 1024\pi cm^3$
find r if minimum sheet metal is to be used



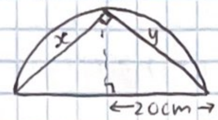
$V = 1024\pi cm^3$
$A = 2\pi rh + 2\pi r^2$
$V = \pi r^2 h$
$\therefore 1024\pi = \pi r^2 h$
$1024 = r^2 h$
$h = \frac{1024}{r^2}$

plug into A formula
$2\pi r \times \frac{1024}{r^2} + 2\pi r^2$
$= 2\pi \frac{1024}{r} + 2\pi r^2$
$A = 2\pi\left(\frac{-1024}{r^2} + 2r\right)$
$= 0$ when $\frac{-1024}{r^2} + 2r = 0$

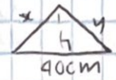
$-1024 + 2r^3 = 0$
$2r^3 = 1024$
$r^3 = 512$
$r = 8$ ← min??

$\frac{d^2A}{dr^2} = 2\pi\left(\frac{2(1024)}{r^3} + 2\right)$
$= > 0 \ \forall$ positive r
$\therefore$ minimum

OPTIMISATION PT 2



① area of triangle $= \frac{1}{2}xy$



② $40^2 = x^2 + y^2$
$y = \sqrt{40^2 - x^2}$

③ sub into eachother
$A = \frac{1}{2}x\sqrt{40^2 - x^2}$
$= \frac{1}{2}x \times (40^2 - x^2)^{1/2}$

④ find dA/dx w/ P.R
$f = 1/2\,x \quad f' = \frac{1}{2}$
$g = (40^2 - x^2)^{1/2} \quad g' = \frac{1}{2}(40^2 - x^2)^{-1/2}$
$dA/dx = \frac{1}{2}(40^2 - x^2)^{1/2} + \frac{1}{2}x \cdot \frac{1}{2}(40^2 - x^2)^{-1/2}(-2x)$
$= \frac{1}{2}(40^2 - x^2)^{1/2} - \frac{1}{2}x^2(40^2 - x^2)^{-1/2}$
$= 0$ when $\frac{1}{2}(40^2 - x^2)^{1/2} - \frac{1}{2}x^2(40^2 - x^2)^{-1/2} = 0$
$\div \frac{1}{2} = (40x^2 - x^2)^{1/2} - x^2(40^2 - x^2)^{-1/2} = 0$
i.e. $(40^2 - x^2)^{1/2} = \frac{x^2}{(40^2 - x^2)^{1/2}} \times (40^2 - x^2)^{1/2}$
40×40
$40^2 - x^2 = x^2$
$40^2 = 2x^2$
$20 \times 40 = x^2$
$x = \sqrt{20 \times 40}$
$x = \sqrt{100 \times 4 \times 2}$
$x = 10 \cdot 2\sqrt{2}$
$x = 20\sqrt{2}$

DISPLACEMENT ETC.

$x(t) = \frac{1}{3}\left(\frac{t^2}{2} - 4\right)^2 - 2$
$0 \leq t \leq 10$

$\boxed{t \text{ @ rest}}$
$\dot{x} = \frac{2}{3}\left(\frac{t^2}{2} - 4\right)(t) = 0$
when $t\left(\frac{t^2}{2} - 4\right) = 0$
i.e. $t^2 = 8 \quad t = 2\sqrt{2}$
or $t = 0$

$\boxed{a \text{ @ } 2s}$
$x = \frac{2}{3}\left(\frac{t^3}{2} - 4(\cdot)\right)$
$x'' = \frac{2}{3}\left(\frac{3t^2}{2} - 4\right)$

$x''\big|_{t=2} = \frac{4}{3}\,as^{-2}$

slowly down since @ $t = 2\sqrt{2}$ it is stationary

SKETCHING

① test for t.p (sign test or $y''(x)$)
e.g. $f(x) = 3 + 2x - x^2$
$f'(x) = 2 - 2x \quad = 0$ @
$x = 1$
$f(1) = 3 + 2 - 1 \quad = 4 \therefore (1, 4)$
$f''(x) = -2 < 0 \quad \therefore$ max

Kyla Orchard

ANTIDIFFERENTIATION

NO PRODUCT, CHAIN OR QUOTIENT RULE

EXAMPLE 1

$\frac{dy}{dx} = \frac{x^4}{4} \quad y = \frac{x^5}{20} + c$

EXAMPLE 2

$\frac{dy}{dx} = \frac{1}{x^4}$

$= x^{-4}$

$y = \frac{-x^{-3}}{3} + c$

EXAMPLE 3

$\frac{dy}{dx} = \sqrt{x}$

$= x^{1/2}$

$y = \frac{x^{3/2}}{3/2} = \frac{2}{3}x^{3/2}$

must match

$$\boxed{\int x^n \, dx \quad y = \frac{x^{n+1}}{n+1} + c \quad n \neq -1 \quad (\text{can't} \div \text{by } 0)}$$

INDEFINITE INTEGRALS

EXAMPLE 4

$\int \sqrt{\frac{1}{4\sqrt{x}}}$

$= \frac{1}{4} \int \frac{dx}{\sqrt{x}}$

$= \frac{1}{4} \int x^{-1/2} \, dx$

$= \frac{1}{4} 2x^{1/2} + c$

$= \frac{1}{2} x^{1/2} + c$

$$\boxed{\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c}$$

RULE 2:

$$\boxed{\int [f(x)]^n f'(x) \, dx = \frac{[f(x)]^{n+1}}{n+1} + c}$$

RULE 2

$y = [f(x)]^{n+1}$

$dy/dx = \int (n+1)[f(x)]^n \cdot f'(x) \, dx$

$y = (n+1) \int [f(x)]^n \cdot f'(x) \, dx$

$[f(x)]^{n+1} = (n+1) \int [f(x)]^n \, f'(x) \, dx$

EXAMPLE 1

$\int 3x \, (x^2+4)^5 \, dx$ — $f(x)$

$= \frac{3}{2} \int [x^2+4]^5 \cdot 2x$

$dx = \frac{3}{2} \frac{(x^2+4)^6}{6} + c$

$= \frac{1}{4}(x^2+4)^6 + c$

EXAMPLE 2

$\int \frac{-3x\sqrt{1-x^2}}{4} \, dx$

$\frac{1}{4} \int (1-x^2)^{1/2} \cdot 3x \, dx$

$\frac{3}{4} \int (1-x^2)^{1/2} \; -x \, dx$ — want a 2 here

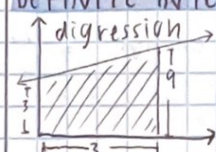
$\frac{3}{4} \cdot \frac{1}{2} \int (1-x^2)^{1/2} \cdot -2x \, dx$

$= \frac{2}{3} \cdot \frac{3}{8} (1-x^2)^{3/2} + c$

$\div 3/2 = x^{2/3}$

$= \frac{1}{4}(1-x^2)^{1/2} + c$

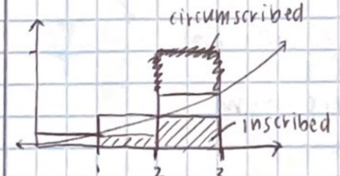
DEFINITE INTEGRALS



$y = 2x+3$

$A = \frac{(9+3)}{2} \times 3$

$= 18 \text{ units}^2$



inscribed	0+2+6	=8
circumscribed	2+6+12	=18
18+8=26	÷2	=13

EXAMPLE

$\int_0^3 x^2 + x \, dx$

$= \left[\frac{x^3}{3} + \frac{x^2}{2}\right]_0^3$

$= \left[\frac{3^3}{3} + \frac{3^2}{2}\right] - \left[\frac{0^3}{3} + \frac{0^2}{2}\right]$

$= 9 + \frac{9}{2}$

$= 27/2 \quad = 13.5$

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a) \quad F = \text{antiderivative of } f}$$

EXAMPLE

determine Area bounded by $y = 9 - x^2$ and the x-axis

$A = \int_{-3}^{3} 9 - x^2 \, dx$

$= 2 \int_0^3 9 - x^2 \, dx$

$= 2\left[9x - \frac{x^3}{3}\right]_0^3$

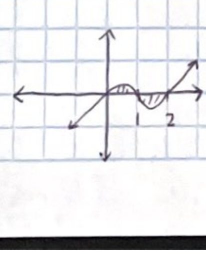
$= 2[27 - 9 - (0)]$

$= 36$

EXAMPLE

$y = x(x-1)(x-2)$

- cubic
- findif
- or



negative to opposite negative area

$A = \int_0^1 x^3 - 3x^2 + 2x \, dx - \int_1^2 x^3 - 3x^2 + 2x \, dx$

$= \left[\frac{x^4}{4} - x^3 + x^2\right]_0^1 - \left[\frac{x^4}{4} - x^3 + x^2\right]_1^2$

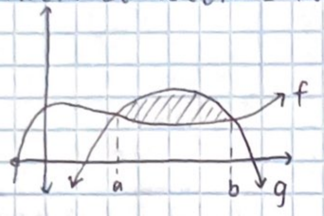
$= \left[\frac{1}{4} \cancel{-1} \cancel{+1} - (0)\right] - [4 - 8 + 4] - \left[\frac{1}{4} \cancel{-1} \cancel{+1}\right]$

$= \frac{1}{4} + \frac{1}{4} \quad = \frac{1}{2}$

on calc: absolute value

$\int_0^2 |x^3 - 3x^2 + 2x| \, dx$

AREA BETWEEN 2 FUNCTIONS



$$\boxed{A = \int_a^b g - f \, dx}$$

displacement $= \int_a^b v(t) \, dt$

displacement ↓
velocity ↓
acceleration

STEPS TO EVALUATE

① integrate $f(x)$ w/ respect to x (omit $+c$)
② sub b into answer from ①
③ sub a into answer from ①
④ (② − ③)

i.e. $\int_a^b f'(x)dx = f(b) - f(a)$

EXAMPLE

$y = x^2 \qquad y = 2x$

intersect when $x^2 = 2x$

$x^2 - 2x = 0 \qquad x(x-2) = 0$

$\therefore x = \underset{a}{0}$ or $x = \underset{b}{2}$

$A = \int_0^2 2x - x^2 \, dx$ change sign from negative

$= \left[x^2 - \frac{x^3}{3}\right]_0^2$

$= 4 - \frac{8}{3} \qquad = \frac{4}{3}$ units^2

SMALL CHANGES

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \approx \frac{\delta y}{\delta x}$$

$$\delta y \approx \frac{dy}{dx} \delta x$$

EXAMPLE

as h $4cm \to 4.1cm$

$A = \frac{bh}{2} = \frac{2h \times h}{2} = h^2$

dA/dh

$\delta A = 2h \, \delta h$

$= 2(4)(0.1)$

$\delta A = 0.8cm^2$

EXAMPLE 2

% error in S/A of sphere of $r=4$, 1% error in radius

volume of sphere

$V = \frac{4}{3}\pi r^3$

$\delta V = {dV}/{dr} \cdot \delta r$

$\frac{\delta V}{V} \times 100 =$ % change in V

$\delta V = 4\pi r^2 \cdot \delta r$

$100 \times \frac{\delta V}{V} = \frac{4\pi r^2 \delta r}{V} \times 100$ sub original V in

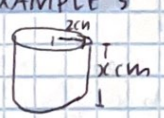
$100 \times \frac{\delta V}{V} = \frac{4\pi r^2 \delta r}{4/3 \pi r^3} \times 100$

$= 3 \frac{\delta r}{r} \times 100$ ← make a 1% (in Qu) $= 1$ 0.01×100

$= 3 \times 1 \qquad \delta V = 3\%$

$c(x) =$ cost, $c'(x) = 1$ more item

EXAMPLE 3



$\delta x = 0.1cm$

$V = \pi r^2 h$

$= \pi(4)x$

$\delta V = 4\pi \delta x$

$= 4\pi(0.1)$

$= 0.4cm^3$

QUADRATIC FORMULA

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$y = [f(x)]^n$

$dy/dx = n[f(x)]^{n-1}(f'(x))$

$y = f(x) \cdot g(x)$

$\frac{dy}{dx} = f'(x) \cdot g(x) + g'(x) f(x)$

$y = \frac{f(x)}{g(x)}$

$= \frac{g(x) \cdot f'(x) - f(x) g'(x)}{[g(x)]^2}$

kyla orchard

~~Fundamental Theorem~~ FUNDAMENTAL THEOREM U3 (chapter 5)

$\int_a^b f(x)\,dx = F(b) - F(a)$ ← derivative of $f(x)$ | $\int_a^b f'(x)\,dx = f(b) - f(a)$ | $\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$

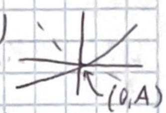
EXAMPLES

$\frac{d}{dx}\left(\int_1^x 4t\,dt\right) = 4x$

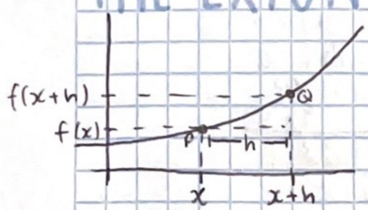
$\frac{d}{dx}\left(\int_0^x 16t(t^2+3)^4\right)$
$= 16x(x^2+3)^4$

e is $\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^n\right] \approx 2.71828$ $\frac{P}{20} = 0.05$ (5% increases)

$A = A_0 e^{kt}$ e.g. for persentages (12%)
$k > 1$ growth $A = A_0 e^{0.12t}$
$k < 1$ decay



THE EXPONENTIAL FUNCTION (U3 chapter 6)



gradient @ P = $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ $y = e^x$ $\frac{dy}{dx} = e^x$

EXAMPLE
$y = e^{x^2-5x+1}$ ∴ $y = e^u$ $\frac{dy}{dx} = (2x-5)e^{x^2-5x+1}$
$u = x^2-5x+1$

$y = e^{f(x)}$ $\frac{dy}{dx} = f'(x)e^{f(x)}$

EXAMPLE
$y = x^2 e^x$
$dy/dx = e^x(2x) + x^2(e^x)$
$= xe^x(2+x)$

$\frac{dP}{dt} = kP$ then $P = P_0 e^{kt}$
$P_0 = P$ when $t=0$

EXAMPLE
$y = \frac{e^x}{x}$ $\frac{dy}{dx} = \frac{xe^x - e^x}{x^2}$
$= \frac{e^x(x-1)}{x^2}$
then @ $dy/dx = 0$, $x = 1$
if $x = 1$, $y = e$ (then graph)

EXAMPLE
if $dP/dt = 0.025P$ $P = P_0 e^{0.025t}$
& $P = 2000$ when $t = 10$ $2000 = P_0 e^{0.025(10)}$
$P_0 = 1557.6$

$\int e^x\,dx = e^x + c$ $\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$

EXAMPLE
$\int 5e^{3x}\,dx$
$= \frac{5}{3}e^{3x} + c$

EXAMPLE
$\int \frac{5}{e^x}\,dx = \frac{e^x \times 0 - 5 \times e^x}{(e^x)^2} = \frac{-5e^x}{(e^x)^2} + c = \frac{-5}{e^x} + c$

CALCULUS OF TRIGONOMETRIC FUNCTIONS U3 ch7

$\lim_{h\to 0} \frac{\sin h}{h}$ | $\lim_{h\to 0} \frac{1-\cos h}{h}$

$y = \sin x$ $y' = \cos x$
$y = \cos x$ $y' = -\sin x$
$\cos x \times \cos x = \cos^2 x$

EXAMPLE
$y = \cos(-3x^2)$
$y' = -\sin(-3x^2)\cdot(-6x)$
$= 6x\sin(-3x^2)$

EXAMPLE
$y = e^{2x}\sin(3x)$
$y' = 2e^{2x}\cdot\sin(3x) + e^{2x}\cdot\overline{\cos(3x)\cdot 3}$ ← y' of sin(3x)

$\int \sin x\,dx = -\cos x + c$
$\int \cos x\,dx = \sin x + c$

EXAMPLE
$\int \cos 2x\,dx$
$= \frac{1}{2}\sin 2x + c$

DON'T DO THIS
~~$\int \cos(x^2)\,dx \neq \frac{\sin(x^2)}{2x} + c$~~

$y = \tan x$
$y' = \frac{1}{\cos^2 x}$

$\int \cos(ax)\,dx$, has to be linear
$= \frac{\sin(ax)}{a} + c$
$= \frac{1}{a}\sin(ax) + c$

EXAMPLE
$\int -8\cos 10x$
$= -\frac{8}{10}\sin 10x + c$

EXAMPLE
$-6\sin\frac{2x}{3}$
$= \frac{3}{2}(-6)\left(-\cos\frac{2x}{3}\right) + c$
$= 9\cos\frac{2x}{3} + c$

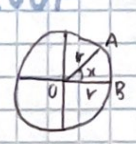
EXAMPLE
$\sin^4 x$
$y' = 4(\sin x)^3\cos x$
$= 4\sin^3 x\cos x$

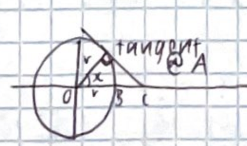
EXAMPLE
$x^2 + \sin x$
$y' = 2x + \cos x$

EXAMPLE
$5\sin x$
$y' = \sin x \times 0 + 5 \times \cos x$
$= 5\cos x$

EXAMPLE
$(2-\cos x)(1+\sin x)$
$y' = (1+\sin x)\times(\sin x) + (2-\cos x)\times\cos x$
$= \sin x + \sin^2 x + 2\cos x - \cos^2 x$

PROOF





$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$
$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$

EXAMPLE
$\int 15\cos 5x\,dx$
$3 \times \int 5\cos 5x\,dx$
$= 3\sin 5x + c$

DRV's
4.3 chapter 8

- probabilities add to 1
- outcomes must be discrete and random
- $P(X=x) \geq 0$

x	cumulative:
$P(X=x)$	$P(X \leq x)$

$s \cdot d^2 = var$ ← δ

$\bar{x} = E(X) = (x \times P(X=x)) + \ldots$ ← expected value

$\mu = \sum x \cdot P(x)$

$var(X) = \sum (x-\mu)^2 P(x)$ ← (standard deviation)^2

$var(X) = E(X^2) - [E(X)]^2$

$\delta = \sqrt{var}$

UNIFORM DISTRIBUTION

all $P(x)$ are the same

$E(X) = \sum_{x=1}^{n} x \times \frac{1}{n} = \frac{n+1}{2}$

$\sum_{\square=\square}^{\square} (\square)$

$var(X) = E(X^2) - \mu^2$

$= \sum_{x=1}^{n} x^2 \times \frac{1}{n} - \left(\frac{n+1}{2}\right)^2$

LINEAR CHANGES

data	mean	s·d
x	μ	δ
$x+c$	$\mu+c$	δ
kx	$k\mu$	$\lvert k \rvert \times \delta$
$kx+c$	$k\mu+c$	$\lvert k \rvert \times \delta$

	faulty	okay
from	5	95
choose	1	3

if $P(X=2)$

$= \frac{95}{100} \times \frac{94}{99} \times \frac{5}{98} \times \frac{4}{97} \times 6$

90 choose 4 $= {}^{90}C_4$

$= {}^{5}C_1 \times {}^{95}C_3$

$P(X=1) = \frac{{}^{5}C_1 \times {}^{95}C_3}{{}^{100}C_4}$

QUOTIENT

$\frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$

PRODUCT

$f'(x)g(x) + g'(x)f(x)$

CHAIN

$\frac{dy}{du} \times \frac{du}{dx}$

$\int e^{f(x)} f'(x)\,dx = e^{f(x)} + c$

INTEGRATION - EXPONENTIALS

if $dy/dx = e^x$

$y = \int e^x dx$

$= e^x + c$

$\int e^{f(x)} \cdot f'(x)\,dx$

$= e^{f(x)} + c$

$\sin(A+B) = \sin A \cos B + \cos A \sin B$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

$\theta = rad \times \frac{180}{\pi}$

$rad = \theta \times \frac{\pi}{180}$

$180° = \pi$

$360° = 2\pi$

$90° = \frac{\pi}{2}$

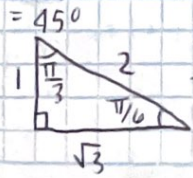
S	A
T	C

$\sin = y$

$\cos = x$

$A = \int_a^b g - f\,dx$

$\pi/4 = 45°$



$\pi/3 = 60°$

$\pi/6 = 30°$

$\sin(0) = 0$

$\cos(0) = 1$

$\sin(180) = 0$

$\cos(180) = -1$

$\sin(90) = 1$

$\cos(90) = 0$

LOGARITHMS

exponential form $\longleftrightarrow$ log form

$a^x = b \longleftrightarrow \log_a b = x$

$\log_{10} 1000 = 3$

$\hookrightarrow 10^3 = 1000$

example:

$\log_2 16 = x$

$2^x = 2^4$

$x = 4$

$\log_2 16 = 4$

LOG LAWS

$\log_a b = x$

$b = a^x$

$a^x a^y = a^{x+y}$

$bc = a^{x+y}$

$\log_a c = y$

$c = a^y$

$a^x \div a^y = a^{x-y}$

$b \div c = a^{x-y}$

$\log_a (bc) = x + y$

$\log_a (bc) = \log_a b + \log_a c$

$\log_a a = 1$

$\log_a 1 = 0$

$\log_a (b^c) = c \log_a b$

$\log_a (\frac{1}{b}) = -\log_a b$

$\log_a b = \dfrac{\log_c b}{\log_c a}$

$\log_a (\frac{b}{c}) = x - y$

$\log_a (\frac{b}{c}) = \log_a b - \log_a c$

NATURAL LOGS

$b = e^x \qquad \ln = \log_e x$

$x = \log_e b$

example:

$2^{5x-1} = 3^x$

$\log(2^{5x-1}) = \log(3^x)$

$(5x-1)\log 2 = (x)\log 3$

$5x \log 2 - \log 2 = x \log 3$

$x(5\log 2 - \log 3) = \log 2$

$x = \dfrac{\log 2}{5\log 2 - \log 3}$

GRAPHS OF LOG FUNCTIONS

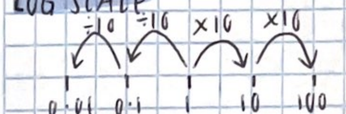
$y = af(x)$ — a = dilation; $-a$ = reflected over x-axis

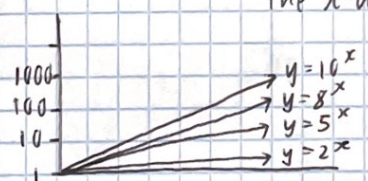
$y = f(x-c)$ — $+c$ = left; c = right

$y = f(x) - d$ — $+d$ = up; $-d$ = down

$y = f(bx)$ — b = dilation of the graph parallel to the x-axis w/ a factor of $\frac{1}{b}$

LOG SCALE





example

$\log_e [x(x+3)]$

$= \ln(x^2 + 3x)$

$\frac{dy}{dx} = \dfrac{1}{x^2+3x} \cdot (2x+3)$

$= \dfrac{2x+3}{x^2+3x} = \dfrac{2x+3}{x(x+3)}$

DIFFERENTIATION

$y = \log_e x$

$x = e^y \qquad dy/dx = e^y$

$\therefore dy/dx = \frac{1}{e^y} = \frac{1}{x}$

$y = \log_e f(x)$

$dy/dx = \dfrac{f'(x)}{f(x)}$

example $e^x \log_e x$

$f = e^x \qquad f' = e^x$

$g = \log_e x \qquad g' = \frac{1}{x}$

$\frac{dy}{dx} = (e^x)(\log_e x) + (\frac{1}{x})(e^x)$

$= e^x \log_e x + \frac{e^x}{x}$

INTEGRATION

$\int \frac{1}{x}\, dx$

$= \ln x + c$

$\int \dfrac{f'(x)}{f(x)}\, dx$

$= \ln f(x) + c$

for $f(x) > 0$

example $\log_e (\frac{1}{x})$

$= -\log_e x \qquad = -\frac{1}{x}$

example

$\dfrac{\log_e x}{x} \qquad f = \log_e x \quad f' = \frac{1}{x}$

$g = x \quad g' = 1$

$\frac{dy}{dx} = \dfrac{(x)(\frac{1}{x}) - (\log_e x)(1)}{(x)^2}$

$= \dfrac{\frac{x}{x} - \log_e x}{(x)^2}$

$= \dfrac{1 - \log_e x}{(x)^2}$

example

$\log_e [(x^2+5)^4]$

$y = \log_e u \qquad u = (x^2+5)^4$

$\frac{dy}{du} = \frac{1}{u} \qquad \frac{du}{dx} = 4(x^2+5)^3 \cdot (2x)$

$dy/dx = \frac{1}{u} \cdot 4(x^2+5)^3\, 2x$

$= \dfrac{8x\,\cancel{(x^2+5)^3}}{(x^2+5)^{4\,1}}$

$= \dfrac{8x}{x^2+5}$

example $y = x\log_e x$ @ $(e^2, 2e^2)$

$f = x \qquad f' = 1$

$g = \log_e x \qquad g' = \frac{1}{x}$

$\frac{dy}{dx} = (1)(\log_e x) + (\frac{1}{x})(x)$

$= \log_e x + 1 \qquad$ sub in $x = e^2$

$= \log_e(e^2) + 1$

$= 2 + 1 \qquad = 3$

example

$\int \frac{5}{x}dx = 5\ln x + c$

example

$\int (x + \frac{2}{x})\,dx$

$= \frac{x^2}{2} + 2\ln x + c$

example

$(\frac{1}{2})\int \frac{(2)1}{2x}\,dx$

$= \frac{1}{2}\ln x + c$

example

$\int (x^2 + \frac{5}{x})\,dx$

$= \frac{x^3}{3} + 5\ln x + c$

example

$5\int \frac{\frac{1}{2}\,10}{2x+1}\,dx$ ↪ since $f' = 2$

$= 5\ln(2x+1) + c$

example

$3\int \frac{\frac{1}{3}6x+15}{x^2+5x}$ ↪ $f' = 2x+5$

$= 3\ln(x^2+5x) + c$

example

$\int \frac{e^x+1}{e^x+x}$

$= \ln(e^x+x) + c$

example

$\int_1^3 \frac{1}{x}\,dx$

$[\ln x]_1^3$

$= [\ln 3] - [\underbrace{\ln 1}_{0}]$

$= \ln 3$

example

$2\log 3 + \log 16 - 2\log(\frac{6}{5})$

$= \log 3^2 + \log 16 - \log(\frac{6}{5})^2$

$= \log\left(\frac{3^2 \times 16}{(\frac{6}{5})^2}\right)$

$= \log\left(\frac{9 \times 16}{(36/25)}\right)$

$= \log(4 \times 25)$

$= \log 100 \quad = 2$

example

$\log_3 45 = \log(9 \times 5)$

$= \log(3^2 \times 5)$

$= \log 3^2 + \log 5$

$= 2\log 3 + \log 5$

example

$\log\left(\frac{x^2 y^{1/2}}{z^3}\right)$

$= \log(x^2) + \log(y^{1/2}) - \log(z)^3$

$= 2\log x + \frac{1}{2}\log y - 3\log z$

$= 2a + \frac{b}{2} - 3c$

example

$\frac{\log_6 8}{\log_6 32} = \frac{\log_6 2^3}{\log_6 2^5} = \frac{3\cancel{\log_6 2}}{5\cancel{\log_6 2}} = \frac{3}{5}$

example

$\log_2 0.12 = \log_2 \frac{12}{100}$

$= \log_2 (3/25)$

$= \log_2 3 - \log_2 5^2$

example

$k = \int_2^6 \frac{2}{x}$

$= [2\ln x]_2^6$

$= 2\ln 6 - 2\ln 2$

$= 2(\ln(\frac{6}{2})) = 2\ln 3 \quad = \ln 3^2$

$= \ln 9 \quad k = \ln 9$

$e^k = 9$ ↩ log → exponential form

$\theta = \text{rad} \times \frac{180}{\pi}$

$\text{rad} = \theta \times \frac{\pi}{180}$

$180° = \pi$

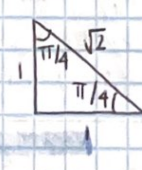
$360° = 2\pi$

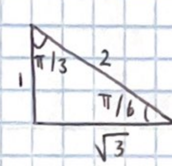
$90 = \pi/2$

$30° = \pi/6$

$60° = \pi/3$

$45° = \pi/4$





S	A
T	C

$\sin(0°) = 0$

$\sin(90°) = 1$

$\sin(\pi/2) = 1$

$\sin(180°) = 0$

$\sin(\pi) = 0$

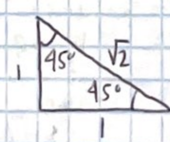
$\cos(0°) = 1$

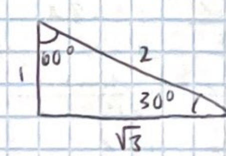
$\cos(90°) = 0$

$\cos(\pi/2) = 0$

$\cos(180°) = -1$

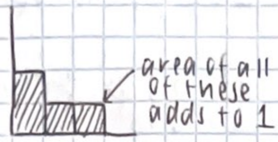
$\cos(\pi) = -1$





~~CRV~~'s

DISTRIBUTION AS A HISTOGRAM



EXPECTED VALUE & VARIANCE

$E(x) = \mu = \int_{-\infty}^{\infty} x\, p(x)\, dx$ ← $f(x)$

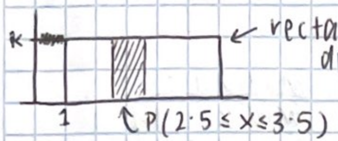
$var = \delta^2 = \int_{-\infty}^{\infty} (x-\mu)^2 p(x)\, dx$

PROBABILITY DENSITY FUNCTION (pdf)

$P(a \le x \le b) = \int_a^b p(x)\, dx$

$P(x=a)$ is negligible

$P(x \geq a) = P(x > a)$

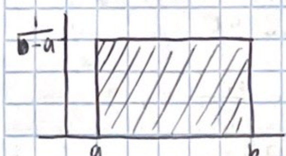


$k = 0{\cdot}2$ because area $= 1$ $k(5) = 1$

$\begin{cases} 0{\cdot}2 & \text{for } 1 \le x \le 6 \\ 0 & \text{otherwise} \end{cases}$

- cannot dip below the x-axis because that = negative probability

UNIFORM DISTRIBUTION

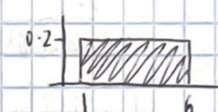


$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$

mean/expected value = halfway between a and b

i.e. $\frac{a+b}{2}$

CUMULATIVE DISTRIBUTION FUNCTION



for $1 \le k \le 6$

$P(x \le k) = \int_1^k 0{\cdot}2\, dx$

$= [0{\cdot}2x]_1^k$

$\therefore \begin{cases} 0 & \text{for } x < 1 \\ 0{\cdot}2(x-1) & \text{for } 1 \le x \le 6 \\ 1 & \text{for } x > 6 \end{cases}$ $= 0{\cdot}2(k-1)$

CDF formula

NON-UNIFORM DISTRIBUTION

$f(x) \geq 0$ for all x in $a < x < b$

$1 = \int_a^b f(x)\, dx$

$f(x) = \begin{cases} ke^{-kx} & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$

if the random variable X has mean μ and standard deviation δ (var δ^2) then the random variable $aX+b$ will have mean $a\mu+b$ and standard deviation $|a|\delta$ (and variance $a^2\delta^2$)

example

$f(x) = \frac{x}{2}$ for $0 \le x \le 2$

$E(X) = \frac{1}{2}\int_0^2 x \times x\, dx$

$= \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2 = \frac{4}{3}$

$E(X^2) = \frac{1}{2}\int_0^2 x^2 \times x\, dx$

$= \frac{1}{2}\left[\frac{x^4}{4}\right]_0^2 = 2$

$var(X) = 2 - (4/3)^2 = 2/9$

example

$f(x) = \frac{2(x+1)}{3}$ for $0 \le x \le 1$

$E(X) = \frac{2}{3}\int_0^1 x \times (x+1)\, dx$

$= \frac{2}{3}\left[\frac{x^3}{3} + \frac{x^2}{2}\right]_0^1 = \frac{5}{9}$

$E(X^2) = \frac{2}{3}\int_0^1 x^2 \times (x+1)\, dx$

$= \frac{2}{3}\left[\frac{x^4}{4} + \frac{x^3}{3}\right]_0^1 = \frac{7}{18}$

$var(X) = 7/18 - (5/9)^2 = 13/162$

example

random variable X has mean $2/3$ and $f(x) = kx$ for $0 \le x \le a$

find a and k

$\int_0^a kx\, dx = 1$

$\left[\frac{kx^2}{2}\right]_0^a = 1$ → $ka^2 = 2$

$E(X) = \int_0^a x \times kx\, dx = 2/3$

$\left[\frac{kx^3}{3}\right]_0^a = \frac{2}{3}$ $ka^3 = 2$

$\begin{cases} ka^2 = 2 \\ ka^3 = 2 \end{cases}$ $k = 2$ $a = 1$

example

$f(x) = kx^3$ for $0 \le x \le 1$

find k

$k \int_0^1 x^3 \, dx$

$k \times 1/4 = 1$

$k = 4$

find $P(X \le 0.6 \mid X \le 0.8)$

$P(X \le 0.6 \mid X \le 0.8) = \dfrac{P(X \le 0.6 \cap X \le 0.8)}{P(X \le 0.8)}$

$= \dfrac{P(X \le 0.6)}{P(X \le 0.8)}$

$= \dfrac{0.1296}{0.4096} = 0.3164$

determine μ & σ^2

$\mu = 4 \int_0^1 x \times x^3 \, dx$

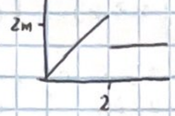
$= 4/5$

$\sigma^2 = 4 \int_0^1 (x - \frac{4}{5})^2 \times x^3 \, dx$

$= 2/75$

example

$f(t) = \begin{cases} mt & 0 \le t \le 2 \\ \frac{1}{4} & 2 < t < 4 \end{cases}$



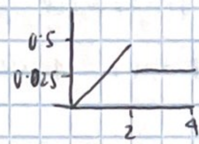
find m:

area under curve = 1

$(\frac{1}{2} \times 2 \times 2m) + (2 + \frac{1}{4}) = 1$

$m = 1/4$

sketch



find $P(T \le 1)$

$= \frac{1}{2} \times 1 \times \frac{1}{4}$

$= 1/8$

median of T

median = m

$P(X \le m) = 0.5$

$P(X \le 2) = 0.5$

$\therefore$ median of $T = 2$

example

$f(x) = x^2 + ax$ for $0 < x < 1$

find a

$\int_0^1 x^2 dx + a \int_0^1 x \, dx = 1$

$\frac{1}{3} + \frac{a}{2} = 1$ $a = 4/3$

find $P(X > 0.5)$

$P(X > 0.5) = \int_{0.5}^1 x^2 + \frac{4x}{3} \, dx$

$= \frac{19}{24}$

find k if $P(X \le k) = 0.9$

$\int_0^k x^2 + \frac{4x}{3} \, dx = 0.9$

$= \frac{k^3}{3} + \frac{2k^2}{3} = 0.9$

$k = 0.9558$

mean & variance

$E(X) = \int_0^1 x \times (x^2 + \frac{4x}{3}) \, dx$

$= 25/36$

$var(X) = \int_0^1 (x - \frac{25}{36})^2 \times (x^2 + \frac{4x}{3}) \, dx$

$= 331/6480$

example find k

area under = 1

hence, $(k \times 1) + \frac{1}{2} \times 1 \times k = 1$

$k = 2/3$

$P(X < 0.5)$

$= 1 - \frac{1}{2} \times \frac{2}{3}$

$= 2/3$

$P(X > 0 \mid X < 0.5)$

$= \dfrac{P(X > 0 \cap X < 0.5)}{P(X < 0.5)}$

$= \dfrac{P(0 < X < 0.5)}{P(X < 0.5)} = \dfrac{1/3}{2/3} = \dfrac{1}{2}$

find a if $P(X > a) = 1/2$

area of rectangle to the right of the line $x = a$ is $(1 - a) \times 2/3$

$\therefore (1 - a) \times 2/3 = 1/2$

$a = 1/4$

example

$f(x) = k(\sqrt{4 - x})$ for $0 \le x \le 4$

where k is a real constant

show $k = 3/16$

$k \int_0^4 \sqrt{4 - x} \, dx = 1$

$k \times \frac{16}{3} = 1$ $k = \frac{3}{16}$

$P(X > 1 \mid X < 3)$

$= \dfrac{P(X > 1 \cap X < 3)}{P(X < 3)}$

$= \dfrac{P(1 < X < 3)}{P(X < 3)}$

$= 0.5995$

median

$P(X \le m) = 0.5$

$\frac{3}{16} \int_0^m \sqrt{4 - x} \, dx = 0.5$

$\frac{3}{16} \times \left[\frac{2(4 - x)^{3/2}}{-3} \right]_0^m = 0.5$

$\frac{3}{16} \times \left[\frac{2(4 - m)^{3/2}}{-3} - \frac{2(4)^{3/2}}{-3} \right] = 0.5$

$m = 1.48$

example

$f(x) = \begin{cases} 0.5x & 0 \le x < 1 \\ \frac{-x}{6} + \frac{2}{3} & 1 \le x \le k \end{cases}$

show that $k = 4$

$f(1) = \frac{1}{2}$

$\frac{1}{2} \times (\frac{1}{2} \times k) = 1$

$k = 4$

$P(X > 0.5)$

$= 1 - 1/2 \times (1/2 \times 1/4)$

$= 15/16$

$P(X \le 2 \mid X > 0.5)$

$= \dfrac{P(0.5 < X \le 2)}{P(X > 0.5)}$

$= \dfrac{1 - 1/2(1/2 \times 1/4) - 1/2(2 \times 1/3)}{15/16}$

$= 29/45$

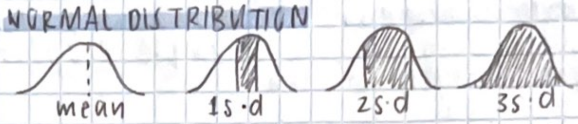
normal distribution

STANDARD SCORES

$$\text{standard score} = \frac{\text{raw score} - \text{mean}}{\text{standard deviation}}$$

z, x

NORMAL DISTRIBUTION



68·3% 95·4% 99·7%

68 : 95 : 99·7 rule

$X \sim N(\mu, \sigma^2)$

σ^2 = variance

μ = mean

$P(A \le X \le B)$ normCDF(A, B, σ, μ)

find k given $P(X \le k)$

invNormCDF(tail setting, $P(X \le k)$, σ, μ)

a% of data lies below the a^{th} percentile

$P(X < k_a) = a$ a = percentile $0 < a < 1$

example:

$X \sim N(20, 5^2)$, find x

z score = 1·5

$z = \frac{x - \mu}{\sigma}$ $1.5 = \frac{x - 20}{5}$

$x = 27.5$

example

find 67^{th} percentile

find the value of k such that $P(X < k) = 0.67$

k = invNormCDF(left, 0·67, 5, 20)

= 22·2

example

$P(X < 21 \mid X > 16)$

$= \frac{P(16 < X < 21)}{P(X > 16)}$

$= \frac{nCDF(16, 21, 5, 20)}{nCDF(16, \infty, 5, 20)} = 0.466$

example

$P(X > k) = 0.75$

k = invNormCDF(right, 0·75, 5, 20)

= 16·63

example

$X \sim N(\mu, \sigma^2)$, mean is twice the variance

$P(X > 10) = 0.3$

$\mu = 2\sigma^2$ $X \sim N(2\sigma^2, \sigma^2)$

invNorm CDF("L", 0·3, 1, 0)

= 0·5244

$z = \frac{x - \mu}{\sigma}$ $0.5244 = \frac{10 - 2\sigma^2}{\sigma}$

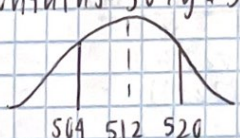
$\sigma = 2.11$ $\mu = 8.89$

example

500g breakfast cereal is normally distributed

$\mu = 512g$ $\sigma = 8g$

a) determine prob. that a randomly chosen box contains 504g & 520g



504 = 1 s·d· below

520 = 1 s·d· above

P = 0·68 (from the rule)

b) determine $P(X < 500g)$

= 0·0668

c) in a sample of 100 boxes, how many boxes should be expected to contain <500g

$P(X < 500) \approx 0.07$

∴ in 100 sample, 7 boxes

example

if 0·7 of the distribution is below 55 then 55 is the 0·7 quantile or 55 is in the 70^{th} percentile

$X \sim N(20, 3^2)$

a) 0·82 quantile

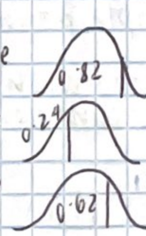
= 22·7

b) 0·24 quantile

= 17·9

c) 0·62 quantile

= 20·9



sampling

RANDOM SAMPLING

randInt (1, 80, 5)
(does 5 integers between 1 and 80)

example

spinner was spun 200 times an A occurred on 43 occasions

a) value of p
20% of the time $p = 0.2$

b) value of $\hat{p}$
$43/200 = 0.215$

c) calculate the mean & s.d. for $\hat{p}$ for such samples of 200 spins
$\hat{p}$ has mean $= p$
s.d of $\sqrt{\frac{p(1-p)}{n}}$ with $p = 0.2$, $n = 200$
$\therefore \hat{p} = 0.2$, $s.d = 0.0283$

types of sampling bias

- selection bias: issues w/ sampling
  - undercoverage: members not adequately represented
  - nonresponse: views of non-respondants are missed due to unwillingness/inability
  - voluntary response: sampling people who are only willing
- response bias: issues w/ surveying
  - leading Q: persuades a response
  - loaded Q: too much information

reducing sampling error:

- ↑ sample size
- exercise true random sampling methods
  - systematic: select every n^{th} item
  - stratified: sample groups that reflect size of same groups in entire population

- actual proportion in a population is fixed
- proportion of people in a sample $\hat{p}$, varies per sample, $\hat{p}$ = random variable
- normally distributed with $\mu = p$, $s.d = \sqrt{\frac{p(1-p)}{n}}$ n = sample size

CONFIDENCE INTEGRALS

$$\left(\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right) = (CI_L, CI_U)$$

$$E = z\sigma = z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$E \propto z$, $E \propto \frac{1}{\sqrt{n}}$

$CI = \hat{p} \pm E$

$$p = \frac{CI_L + CI_U}{2} \qquad E = \frac{CI_U - CI_L}{2}$$

① $p = \frac{CI_L + CI_U}{2}$
② $E = CI_U - p$ $E = p + CI_L$
③ $E_{new} = z_{new}/z_{old} \times E_{old}$
④ new $CI = p \pm E_{new}$

population proportion - p
sample proportion - $\hat{p}$

MARGIN OF ERROR

$$k\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} =$$

example

of 221 50-60 y/o males, 124 had back problems

a) calculate $\hat{p}$
$\therefore \hat{p} = \frac{124}{221} = 0.561$

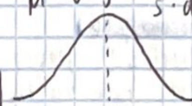
	z score
99%	2.58
95%	1.96
90%	1.645

b) estimate s.d. of $\hat{p}$

$$\approx \sqrt{\frac{\frac{124}{221}\left(1 - \frac{124}{221}\right)}{221}} \approx 0.0334$$

example:

roll a dice 80 times, even = 50%.
$\mu = 0.5$ $s.d. = \sqrt{\frac{0.5(1-0.5)}{80}} \approx 0.0559$



0.5

if $\hat{p}$ is within 2δ's of the mean then
$0.5 - 2 \times 0.0559 < \hat{p} < 0.5 + 2 \times 0.0559$
$0.3882 < \hat{p} < 0.6118$
an approx. 95% of the occasions that we roll a normal 6-sided die 80 times ∴ expect the proportion of even numbers between 39% and 61%

example

54% = μ of whole population
500 people from a community are surveyed
80% chance that in a sample of 500 people occurs between A% & B%.
$\mu = 0.54$
$s.d = \sqrt{\frac{0.54(1-0.54)}{500}} \approx 0.0223$

for $X \sim N(0.54, 0.0223^2)$
if $P(X < k) = 0.1$
$k = 0.5114$
if $P(X > k) = 0.9$
$k = 0.5686$

tail	centre	x_1 invN	0.5114214
prob	0.8	x_2 invN	0.5685786
δ	0.0223	prob	0.8
μ	0.54	δ	0.0223
		μ	0.54

example:

90% CI is (0.38, 0.45)
determine 95%
$p = 0.415$
$E = 0.035$
$E_{new} = 0.0417$